

# On Chebyshev type Inequalities using Generalized k-Fractional Integral Operator

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## Abstract

In this paper, using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function), we establish new results on generalized k-fractional integral inequalities by considering the extended Chebyshev functional in case of synchronous function and some other inequalities.

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**Mathematics Subject Classification :** 26D10, 26A33.

## 1 Introduction

In recent years, many authors have worked on fractional integral inequalities by using different fractional integral operator such as Riemann-Liouville, Hadamard, Saigo and Erdelyi-Kober, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15]. In [12] S. Kilinc and H. Yildirim establish new generalized k-fractional integral inequalities involving Gauss hypergeometric function related to Chebyshev functional. In [5, 10] authors gave the following fractional integral inequalities, using the Hadamard and Riemann-Liouville fractional integral for extended Chebyshev functional.

**Theorem 1.1** *Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ , and  $r, p, q : [0, \infty) \rightarrow [0, \infty)$ . Then for all  $t > 0$ ,  $\alpha > 0$ , we have*

$$\begin{aligned} & 2 {}_H D_{1,t}^{-\alpha} r(t) \left[ {}_H D_{1,t}^{-\alpha} p(t) {}_H D_{1,t}^{-\alpha} (qfg)(t) + {}_H D_{1,t}^{-\alpha} q(t) {}_H D_{1,t}^{-\alpha} (pfg)(t) \right] + \\ & 2 {}_H D_{1,t}^{-\alpha} p(t) {}_H D_{1,t}^{-\alpha} q(t) {}_H D_{1,t}^{-\alpha} (rfg)(t) \geq \\ & {}_H D_{1,t}^{-\alpha} r(t) \left[ {}_H D_{1,t}^{-\alpha} (pf)(t) {}_H D_{1,t}^{-\alpha} (qg)(t) + {}_H D_{1,t}^{-\alpha} (qf)(t) {}_H D_{1,t}^{-\alpha} (pg)(t) \right] + \quad (1.1) \\ & {}_H D_{1,t}^{-\alpha} p(t) \left[ {}_H D_{1,t}^{-\alpha} (rf)(t) {}_H D_{1,t}^{-\alpha} (qg)(t) + {}_H D_{1,t}^{-\alpha} (qf)(t) {}_H D_{1,t}^{-\alpha} (rg)(t) \right] + \\ & {}_H D_{1,t}^{-\alpha} q(t) \left[ {}_H D_{1,t}^{-\alpha} (rf)(t) {}_H D_{1,t}^{-\alpha} (pg)(t) + {}_H D_{1,t}^{-\alpha} (pf)(t) {}_H D_{1,t}^{-\alpha} (rg)(t) \right] \end{aligned}$$

**Theorem 1.2** *Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ , and  $r, p, q : [0, \infty) \rightarrow [0, \infty)$ . Then for all  $t > 0$ ,  $\alpha > 0$ , we have:*

$$\begin{aligned}
& {}_H D_{1,t}^{-\alpha} r(t) \times \\
& \left[ {}_H D_{1,t}^{-\alpha} q(t) {}_H D_{1,t}^{-\beta} (pfg)(t) + 2 {}_H D_{1,t}^{-\alpha} p(t) {}_H D_{1,t}^{-\beta} (qfg)(t) + {}_H D_{1,t}^{-\beta} q(t) {}_H D_{1,t}^{-\alpha} (pfg)(t) \right] \\
& + \left[ {}_H D_{1,t}^{-\alpha} p(t) {}_H D_{1,t}^{-\beta} q(t) + {}_H D_{1,t}^{-\beta} p(t) {}_H D_{1,t}^{-\alpha} q(t) \right] {}_H D_{1,t}^{-\alpha} (rfg)(t) \geq \\
& {}_H D_{1,t}^{-\alpha} r(t) \left[ {}_H D_{1,t}^{-\alpha} (pf)(t) {}_H D_{1,t}^{-\beta} (qg)(t) + {}_H D_{1,t}^{-\beta} (qf)(t) {}_H D_{1,t}^{-\alpha} (pg)(t) \right] + \\
& {}_H D_{1,t}^{-\alpha} p(t) \left[ {}_H D_{1,t}^{-\alpha} (rf)(t) {}_H D_{1,t}^{-\beta} (qg)(t) + {}_H D_{1,t}^{-\beta} (qf)(t) {}_H D_{1,t}^{-\alpha} (rg)(t) \right] + \\
& {}_H D_{1,t}^{-\alpha} q(t) \left[ {}_H D_{1,t}^{-\alpha} (rf)(t) {}_H D_{1,t}^{-\beta} (pg)(t) + {}_H D_{1,t}^{-\beta} (pf)(t) {}_H D_{1,t}^{-\alpha} (rg)(t) \right].
\end{aligned} \tag{1.2}$$

The main objective of this paper is to establish some Chebyshev type inequalities and some other inequalities using generalized k-fractional integral operator. The paper has been organized as follows. In Section 2, we define basic definitions related to generalized k-fractional integral operator. In section 3, we obtain Chebyshev type inequalities using generalized k-fractional. In Section 4, we prove some inequalities for positive continuous functions.

## 2 Preliminaries

In this section, we present some definitions which will be used later discussion.

**Definition 2.1** *Two function  $f$  and  $g$  are said to synchronous (asynchronous) on  $[a, b]$ , if*

$$((f(u) - f(v))(g(u) - g(v))) \geq (\leq) 0, \tag{2.1}$$

for all  $u, v \in [0, \infty)$ .

**Definition 2.2** *[12, 15] The function  $f(x)$ , for all  $x > 0$  is said to be in the  $L_{p,k}[0, \infty)$ , if*

$$L_{p,k}[0, \infty) = \left\{ f : \|f\|_{L_{p,k}[0,\infty)} = \left( \int_0^\infty |f(x)|^p x^k dx \right)^{\frac{1}{p}} < \infty \ 1 \leq p < \infty \ k \geq 0 \right\}, \tag{2.2}$$

**Definition 2.3** [12, 14, 15] Let  $f \in L_{1,k}[0, \infty)$ . The generalized Riemann-Liouville fractional integral  $I^{\alpha,k}f(x)$  of order  $\alpha, k \geq 0$  is defined by

$$I^{\alpha,k}f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt. \quad (2.3)$$

**Definition 2.4** [12, 15] Let  $k \geq 0, \alpha > 0, \mu > -1$  and  $\beta, \eta \in R$ . The generalized  $k$ -fractional integral  $I_{t,k}^{\alpha,\beta,\eta,\mu}$  (in terms of the Gauss hypergeometric function) of order  $\alpha$  for real-valued continuous function  $f(t)$  is defined by

$$I_{t,k}^{\alpha,\beta,\eta,\mu}[f(t)] = \frac{(k+1)^{\mu+\beta+1} t^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^t \tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha-1} \times {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{t})^{k+1}) \tau^k f(\tau) d\tau. \quad (2.4)$$

where, the function  ${}_2F_1(-)$  in the right-hand side of (2.4) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (2.5)$$

and  $(a)_n$  is the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.$$

Consider the function

$$\begin{aligned} F(t, \tau) &= \frac{(k+1)^{\mu+\beta+1} t^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \tau^{(k+1)\mu} \\ &\quad (t^{k+1} - \tau^{k+1})^{\alpha-1} \times {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{t})^{k+1}) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha + \beta + \mu)_n (-n)_n}{\Gamma(\alpha + n) n!} t^{(k+1)(-\alpha-\beta-2\mu-\eta)} \tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha-1+n} (k+1)^{\mu+\beta+1} \\ &= \frac{\tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha-1} (k+1)^{\mu+\beta+1}}{t^{k+1}(\alpha + \beta + 2\mu)\Gamma(\alpha)} + \\ &\quad \frac{\tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha} (k+1)^{\mu+\beta+1} (\alpha + \beta + \mu)(-n)}{t^{k+1}(\alpha + \beta + 2\mu + 1)\Gamma(\alpha + 1)} + \\ &\quad \frac{\tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha+1} (k+1)^{\mu+\beta+1} (\alpha + \beta + \mu)(\alpha + \beta + \mu + 1)(-n)(-n+1)}{t^{k+1}(\alpha + \beta + 2\mu + 1)\Gamma(\alpha + 2)2!} + \dots \end{aligned} \quad (2.6)$$

It is clear that  $F(t, \tau)$  is positive because for all  $\tau \in (0, t)$ ,  $(t > 0)$  since each term of the (2.6) is positive.

### 3 Fractional Integral Inequalities for Extended Chebyshev Functional

In this section, we establish some Chebyshev type fractional integral inequalities by using the generalized  $k$ -fractional integral (in terms of the Gauss hypergeometric function) operator. The following lemma is used for the our main result.

**Lemma 3.1** *Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ , and  $x, y : [0, \infty) \rightarrow [0, \infty)$  be two nonnegative functions. Then for all  $k \geq 0$ ,  $t > 0$ ,  $\alpha > \max\{0, -\beta - \mu\}$ ,  $\beta < 1$ ,  $\mu > -1$ ,  $\beta - 1 < \eta < 0$ , we have,*

$$\begin{aligned} I_{t,k}^{\alpha,\beta,\eta,\mu} x(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (yfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} y(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) \geq \\ I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (yg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (yf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t). \end{aligned} \quad (3.1)$$

**Proof:** Since  $f$  and  $g$  are synchronous on  $[0, \infty[$  for all  $\tau \geq 0$ ,  $\rho \geq 0$ , we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \quad (3.2)$$

From (3.2),

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \quad (3.3)$$

Now, multiplying both side of (3.3) by  $\tau^k x(\tau) F(t, \tau)$ ,  $\tau \in (0, t)$ ,  $t > 0$ . Then the integrating resulting identity with respect to  $\tau$  from 0 to  $t$ , we obtain by definition (2.4)

$$\begin{aligned} I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) + f(\rho)g(\rho) I_{t,k}^{\alpha,\beta,\eta,\mu} (x)(t) \\ I_{t,k}^{\alpha,\beta,\eta,\mu} (yg)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) + f(\rho) I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t). \end{aligned} \quad (3.4)$$

Now, multiplying both side of (3.4) by  $\rho^k y(\rho) F(t, \rho)$ ,  $\rho \in (0, t)$ ,  $t > 0$ , where  $F(t, \rho)$  defined in view of (2.6). Then the integrating resulting identity with respect to  $\rho$  from 0 to  $t$ , we obtain by definition (2.4)

$$\begin{aligned} I_{t,k}^{\alpha,\beta,\eta,\mu} y(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (yfg)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (x)(t) \\ \geq g(\rho) I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (yf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t). \end{aligned} \quad (3.5)$$

This complete the proof of (3.1)

Now, we gave our main result here.

**Theorem 3.2** *Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ , and  $r, p, q : [0, \infty) \rightarrow [0, \infty)$ . Then for all  $k \geq 0$ ,  $t > 0$ ,  $\alpha > \max\{0, -\beta - \mu\}$ ,*

$\beta < 1$ ,  $\mu > -1$ ,  $\beta - 1 < \eta < 0$ , we have,

$$\begin{aligned}
& 2I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \right] + \\
& 2I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right] + \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right] + \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right]
\end{aligned} \tag{3.6}$$

**Proof:** To prove above theorem, putting  $x = p$ ,  $y = q$ , and using lemma 3.1, we get

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t).
\end{aligned} \tag{3.7}$$

Now, multiplying both side by  $(3.7) I_{t,k}^{\alpha,\beta,\eta,\mu} r(t)$ , we have

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right],
\end{aligned} \tag{3.8}$$

putting  $x = r$ ,  $y = q$ , and using lemma 3.1, we get

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t),
\end{aligned} \tag{3.9}$$

multiplying both side by  $(3.9) I_{t,k}^{\alpha,\beta,\eta,\mu} p(t)$ , we have

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (qg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].
\end{aligned} \tag{3.10}$$

With the same arguments as before, we can write

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].
\end{aligned} \tag{3.11}$$

Adding the inequalities (3.8), (3.10) and (3.11), we get required inequality (3.6).

Here, we give the lemma which is useful to prove our second main result.

**Lemma 3.3** *Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ . and  $x, y : [0, \infty[ \rightarrow [0, \infty[$ . Then for all  $k \geq 0$ ,  $t > 0$ ,  $\alpha > \max\{0, -\beta - \mu\}$ ,  $\gamma > \max\{0, -\delta - v\}$ ,  $\beta, \delta < 1$ ,  $v, \mu > -1$ ,  $\beta - 1 < \eta < 0$ ,  $\delta - 1 < \zeta < 0$ , we have,*

$$\begin{aligned} I_{t,k}^{\alpha,\beta,\eta,\mu} x(t) I_{t,k}^{\gamma,\delta,\zeta,v} (yfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} y(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) \geq \\ I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (yg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (yf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t). \end{aligned} \quad (3.12)$$

**Proof:** Now multiplying both side of (3.4) by

$$\begin{aligned} \frac{(k+1)^{v+\delta+1} t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)} \rho^{(k+1)v} y(\rho) \\ (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1}) \rho^k \end{aligned} \quad (3.13)$$

which remains positive in view of the condition stated in (3.12),  $\rho \in (0, t)$ ,  $t > 0$ , we obtain

$$\begin{aligned} \frac{(k+1)^{v+\delta+1} t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)} \rho^{(k+1)v} y(\rho) \\ (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1}) \rho^k I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) \\ + \frac{(k+1)^{v+\delta+1} t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)} \rho^{(k+1)v} y(\rho) f(\rho) g(\rho) \\ (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1}) \rho^k I_{t,k}^{\alpha,\beta,\eta,\mu} x(t) \geq \\ \frac{(k+1)^{v+\delta+1} t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)} \rho^{(k+1)v} y(\rho) g(\rho) \\ (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1}) \rho^k I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) \\ + \frac{(k+1)^{v+\delta+1} t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)} \rho^{(k+1)v} y(\rho) f(\rho) \\ (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1}) \rho^k I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t), \end{aligned} \quad (3.14)$$

then integrating (3.14) over  $(0, t)$ , we obtain

$$\begin{aligned} I_{t,k}^{\alpha,\beta,\eta,\mu} (xfg)(t) I_{t,k}^{\gamma,\delta,\zeta,v} y(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} x(t) I_{t,k}^{\gamma,\delta,\zeta,v} (yfg)(t) \\ \geq I_{t,k}^{\alpha,\beta,\eta,\mu} (xf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} yg(t) + I_{t,k}^{\alpha,\beta,\eta,\mu} (xg)(t) I_{t,k}^{\gamma,\delta,\zeta,v} yf(t), \end{aligned} \quad (3.15)$$

this ends the proof of inequality (3.12).

**Theorem 3.4** Let  $f$  and  $g$  be two synchronous function on  $[0, \infty[$ , and  $r, p, q : [0, \infty) \rightarrow [0, \infty)$ . Then for all  $t > 0$ ,  $\alpha > 0$ , we have:

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \times \\
& \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\gamma,\delta,\zeta,v} (pfg)(t) + 2I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \right] \\
& + \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,v} q(t) + I_{t,k}^{\gamma,\delta,\zeta,v} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \right] I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right] + \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right] + \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (pg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].
\end{aligned} \tag{3.16}$$

**Proof:** To prove above theorem, putting  $x = p$ ,  $y = q$ , and using lemma 3.3 we get

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t).
\end{aligned} \tag{3.17}$$

Now, multiplying both side by  $(3.17) I_{t,k}^{\alpha,\beta,\eta,\mu} r(t)$ , we have

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right],
\end{aligned} \tag{3.18}$$

putting  $x = r$ ,  $y = q$ , and using lemma 3.3, we get

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t),
\end{aligned} \tag{3.19}$$

multiplying both side by  $(3.19) I_{t,k}^{\alpha,\beta,\eta,\mu} p(t)$ , we have

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].
\end{aligned} \tag{3.20}$$

With the same argument as before, we obtain

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) I_{t,k}^{\gamma,\delta,\zeta,v} (pfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,v} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rfg)(t) \right] \geq \\
& I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (rf)(t) I_{t,k}^{\gamma,\delta,\zeta,v} (pg)(t) + (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].
\end{aligned} \tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we follows the inequality (3.16).

**Remark 3.1** *If  $f, g, r, p$  and  $q$  satisfies the following condition,*

1. *The function  $f$  and  $g$  is asynchronous on  $[0, \infty)$ .*
2. *The function  $r, p, q$  are negative on  $[0, \infty)$ .*
3. *Two of the function  $r, p, q$  are positive and the third is negative on  $[0, \infty)$ .*

*then the inequality 3.6 and 3.16 are reversed.*

## 4 Other fractional integral inequalities

In this section, we proved some fractional integral inequalities for positive and continuous functions which as follows:

**Theorem 4.1** *Suppose that  $f, g$  and  $h$  be three positive and continuous functions on  $[0, \infty[$ , such that*

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0; \quad \tau, \rho \in (0, t) \quad t > 0, \quad (4.1)$$

*and  $x$  be a nonnegative function on  $[0, \infty)$ . Then for all  $k \geq 0, t > 0, \alpha > \max\{0, -\beta - \mu\}, \gamma > \max\{0, -\delta - v\}, \beta, \delta < 1, v, \mu > -1, \beta - 1 < \eta < 0, \delta - 1 < \zeta < 0$ , we have,*

$$\begin{aligned} & I_{t,k}^{\alpha,\beta,\eta,\mu}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xfgh)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xfg)(t) \\ & + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xh)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(x)(t) \\ & \geq I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xgh)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xfh)(t) \\ & + I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(xg)(t). \end{aligned} \quad (4.2)$$

**Proof:** Since  $f, g$  and  $h$  be three positive and continuous functions on  $[0, \infty[$  by (4.1), we can write

$$\begin{aligned} & f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) \\ & \geq f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho). \end{aligned} \quad (4.3)$$

Now, multiplying both side of (4.3) by  $\tau^k x(\tau) F(t, \tau)$ ,  $\tau \in (0, t)$ ,  $t > 0$ . Then the integrating resulting identity with respect to  $\tau$  from 0 to  $t$ , we obtain by



definition (2.4)

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t) + f(\rho)g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}x(t) + g(\tau)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + f(\rho)g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) \geq g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t) + g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t) + f(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\end{aligned} \tag{4.4}$$

Now multiplying both side of (4.4) by

$$\begin{aligned}
& \frac{(k+1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)}\rho^{(k+1)v}x(\rho) \\
& (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1})\rho^k
\end{aligned} \tag{4.5}$$

which remains positive in view of the condition stated in (4.2),  $\rho \in (0, t)$ ,  $t > 0$  and integrating resulting identity with respective  $\rho$  from 0 to  $t$ , we obtain

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}x(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(xfgh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}x(t) \\
& + I_{t,k}^{\gamma,\delta,\zeta,v}(xh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgf)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(xfg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) \\
& \geq I_{t,k}^{\gamma,\delta,\zeta,v}xg(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(xgh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + I_{t,k}^{\gamma,\delta,\zeta,v}(xf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(xfh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\end{aligned} \tag{4.6}$$

which implies the proof inequality 4.2.

Here, we give another inequality which is as follows.

**Theorem 4.2** *Let  $f, g$  and  $h$  be three positive and continuous functions on  $[0, \infty[$ , which satisfying the condition (4.1) and  $x$  and  $y$  be two nonnegative functions on  $[0, \infty)$ . Then for all  $k \geq 0$ ,  $t > 0$ ,  $\alpha > \max\{0, -\beta - \mu\}$ ,  $\gamma > \max\{0, -\delta - v\}$ ,  $\beta, \delta < 1$ ,  $v, \mu > -1$ ,  $\beta - 1 < \eta < 0$ ,  $\delta - 1 < \zeta < 0$ , we have,*

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yfg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yfg)(t) \\
& + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yh)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}y(t) \\
& \geq I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(ygh)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yfh)(t) \\
& + I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}(yg)(t).
\end{aligned} \tag{4.7}$$

**Proof:** Multiplying both side of (4.3) by  $\tau^k x(\tau)F(t, \tau)$ ,  $\tau \in (0, t)$ ,  $t > 0$ , where  $F(t, \tau)$  defined by (2.6). Then the integrating resulting identity with

respect to  $\tau$  from 0 to  $t$ , we obtain by definition (2.4)

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t) + f(\rho)g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}x(t) + g(\tau)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + f(\rho)g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) \geq g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t) + g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t) + f(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\end{aligned} \tag{4.8}$$

Now multiplying both side of (4.8) by

$$\begin{aligned}
& \frac{(k+1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2v)}}{\Gamma(\gamma)}\rho^{(k+1)v}y(\rho) \\
& (t^{k+1} - \rho^{k+1})^{\gamma-1} \times {}_2F_1(\gamma + \delta + v, -\zeta; \gamma; 1 - (\frac{\rho}{t})^{k+1})\rho^k
\end{aligned} \tag{4.9}$$

which remains positive in view of the condition stated in (4.7),  $\rho \in (0, t)$ ,  $t > 0$  and integrating resulting identity with respective  $\rho$  from 0 to  $t$ , we obtain

$$\begin{aligned}
& I_{t,k}^{\alpha,\beta,\eta,\mu}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,v}y(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(yfg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}x(t) \\
& + I_{t,k}^{\gamma,\delta,\zeta,v}(yh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgf)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(yfg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) \\
& \geq I_{t,k}^{\gamma,\delta,\zeta,v}(yg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(ygh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t) \\
& + I_{t,k}^{\gamma,\delta,\zeta,v}(yf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t) + I_{t,k}^{\gamma,\delta,\zeta,v}(yfh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\end{aligned} \tag{4.10}$$

which implies the proof inequality 4.7.

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